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A NOTE ON EXTENSION OF 1D EQUATION GOVERNING BUILDUP OF PORE PRESSURE TO 3D. PRESENCE OF A STRUCTURE

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1. 1D equation

The equation governing the buildup of pore pressure in a progressive wave (Fig. 1), Eq. 3.28 in the book of Sumer (2014), reads

$$\frac{\partial \overline{p}}{\partial t} = c_v \frac{\partial^2 \overline{p}}{\partial z^2} + f \qquad (1)$$

where c_v (Eq. 3.26 in Sumer, 2014), the coefficient of consolidation:

$$c_v = \frac{Gk}{\gamma} \frac{2-2\nu}{(1-2\nu)+(2-2\nu)\dfrac{nG}{K'}} \qquad (2)$$

Here, $\overline{p}$ is the period-averaged pore pressure, i.e., the pore pressure accumulated with the introduction of the wave, z is the distance measured from the seabed with the vertical axis z directed downward, t is the time, G is the shear modulus of the soil, k is the coefficient of permeability, ν is Poisson's ratio, n is the porosity, K' is the apparent bulk modulus of elasticity of water and γ is the specific weight of water. The quantity f, the so-called source term, is the total amount of pore pressure generated per unit time and per unit volume of soil (including the pores). This pressure generation is associated with the shear stress in the soil τ_y (Fig. 1) generated by the wave (see Sumer, 2014, Section 2.2.1), discussed in Section 3 below.

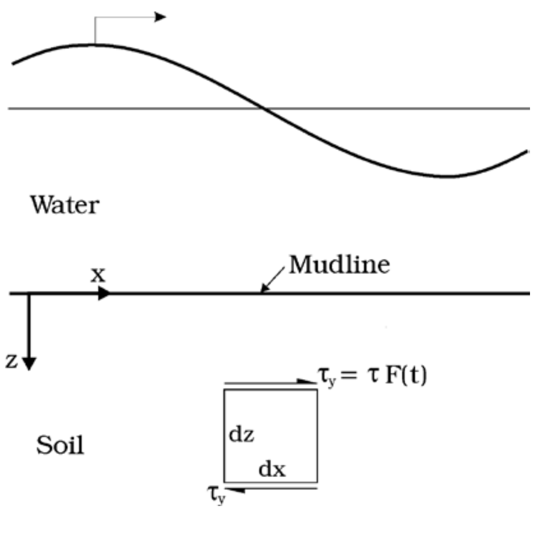

Fig. 1. Definition sketch in 1D.

2. Extension to 3D

In the case when a structure is placed on the seabed, e.g., a gravity structure, the process of buildup of pore pressure will be 3D in the near field. Therefore, the equation governing the buildup of pore pressure in a progressive wave needs to be extended to 3D, to be able to deal with the buildup of pore pressure in the neighbourhood of the structure. Sumer and Cheng (1999) were the first to extend the 1D equation to multi-dimensions (2D in Sumer and Cheng, 1999) where the latter authors studied the buildup of pore pressure around a pipeline buried in the soil in a progressive wave.

Extension of the 1D equation to 3D is straightforward, considering the diffusion analogy (see Sumer, 2014, p. 74):

$$\frac{\partial \overline{p}}{\partial t} = c_{v,x}\frac{\partial^2 \overline{p}}{\partial x^2} + c_{v,y}\frac{\partial^2 \overline{p}}{\partial y^2} + c_{v,z}\frac{\partial^2 \overline{p}}{\partial z^2} + f \qquad (3)$$

with the coefficients of consolidations (analogous to diffusion coefficients in the diffusion analogy)

$$c_{v,x} = \frac{Gk_x}{\gamma}\frac{2-2\nu}{(1-2\nu)+(2-2\nu)\dfrac{nG}{K'}} \qquad (4)$$

and similar expressions for $c_{v,y}$ and $c_{v,z}$.

Even the variations of the shear modulus in the three directions can also be factored in. Nevertheless, the influence of the coefficients of consolidations is influenced most by the permeability, and therefore the above formulation should be adequate!

When the coefficients of consolidation are considered unaltered in the three directions, the 3D equation will read:

$$\frac{\partial \overline{p}}{\partial t} = c_v\left(\frac{\partial^2 \overline{p}}{\partial x^2} + \frac{\partial^2 \overline{p}}{\partial y^2} + \frac{\partial^2 \overline{p}}{\partial z^2}\right) + f \qquad (5)$$

or

$$\frac{\partial \overline{p}}{\partial t} = c_v(\nabla^2 \overline{p}) + f \qquad (6)$$

3. Source term *f* in 1D

The source term f, the total amount of pore pressure generated per unit time and per unit volume of soil (including the pores), i.e., the pore pressure generated in short, is expressed as follows (Eqs. 3.31 and 3.32, Sumer, 2014)

$$f = \frac{\sigma_0'}{N_\ell T} \qquad (7)$$

with N_ℓ:

$$N_\ell = \left(\frac{1}{\alpha} \frac{\tau}{\sigma_0'} \right)^{1/\beta} \qquad (8)$$

or

$$f = \frac{\sigma_0'}{T} \frac{1}{\left(\frac{1}{\alpha} \frac{\tau}{\sigma_0'} \right)^{1/\beta}} \qquad (9)$$

in which T is the wave period, N_ℓ is the number of wave cycles to cause liquefaction, α and β are empirical coefficients which are a function of the relative density of the soil, D_r (see e.g., Sumer 2014, p. 81), given as

$$\alpha = 0.34 D_r + 0.084, \qquad \beta = 0.37 D_r - 0.46 \qquad (10)$$

with the knowledge that the above expressions can be extrapolated to the range $0 < D_r < 1$ (Sumer, 2014, p. 81), and σ_0' is the initial mean normal effective stress (Sumer, 2014, p. 46, Eq. 3.6). *The quantity τ in Eq. (9), is the amplitude of the shear stress in the soil under the "forcing" progressive wave, i.e., the amplitude of τ_y (Fig. 1).*

4. Source term *f* in 3D

Fig. 2 displays the definition sketch. The seabed and the structure are subject to a progressive wave travelling in the x direction. In the near field, there will, in general, be two more shear stress components, namely τ_x and τ_z in addition to τ_y, as illustrated in Fig. 2.b. In reality, however, τ_y and, to a smaller extent, τ_x will be there, but not the third component τ_z unless the structure is subject to a torque in the z direction. Nevertheless, we will include τ_z in our analysis for generality. We note that our analysis also accounts for the rocking motion of the structure around the y axis, induced by the wave.

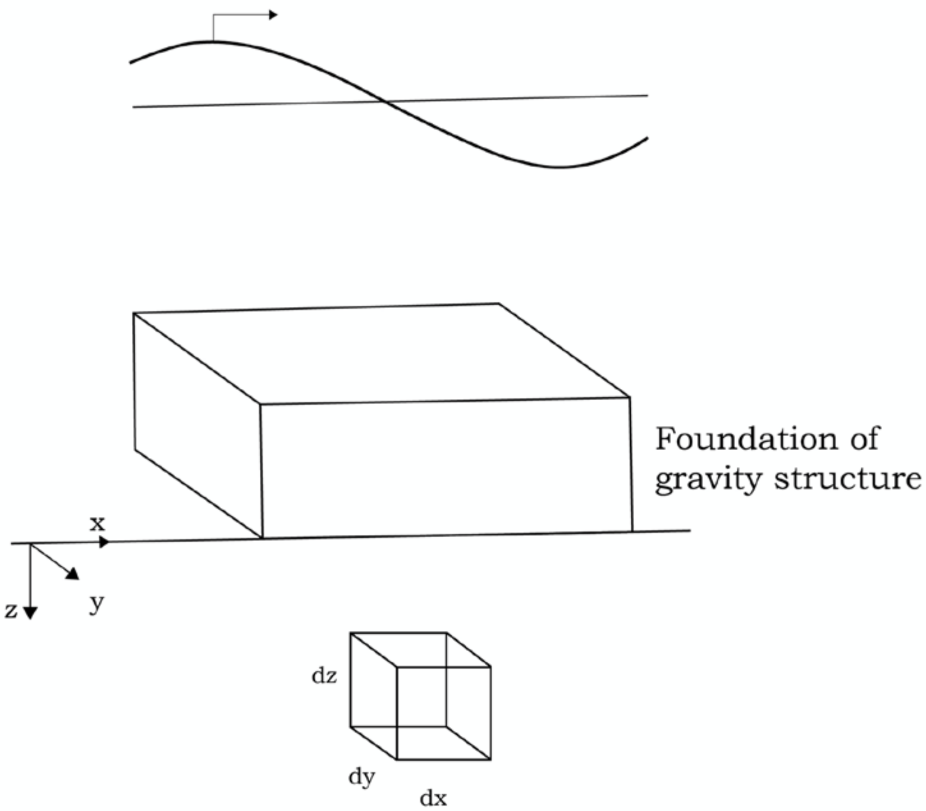

Fig. 2.a. Definition sketch. 3D case with the presence of a gravity structure (with rectangular-prism-shape foundation).

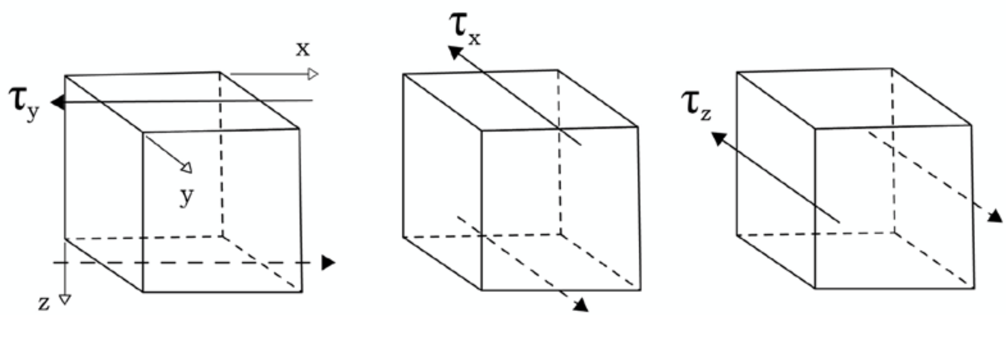

Fig. 2.b. Shear stresses on the sides of the 3D soil element (dx, dy, dz) in Fig. 2.a. The shear stress τ_z is identically equal to zero unless the structure is subject to a torque in the z direction.

First of all, from the previous section (1D case), the pressure generated by τ_y, Eq. (9):

$$\textit{pressure generated by}\,\tau_y = \frac{\sigma_0'}{T}\frac{1}{\left(\frac{1}{\alpha}\frac{|\tau_y|}{\sigma_0'}\right)^{1/\beta}} \qquad (11)$$

in which $|\tau_y|$ is the amplitude of the shear stress τ_y.

We, to a first approximation, assume that the other two components of the shear stress generate pore pressures in a similar way, and therefore the pressures generated by these shear stress components may be written as follows:

$$\textit{pressure generated by}\,\tau_x = \frac{\sigma_0'}{T}\frac{1}{\left(\frac{1}{\alpha}\frac{|\tau_x|}{\sigma_0'}\right)^{1/\beta}} \qquad (12)$$

in which $|\tau_x|$ is the amplitude of the shear stress τ_x; and similarly

$$\textit{pressure generated by}\,\tau_z = \frac{\sigma_0'}{T}\frac{1}{\left(\frac{1}{\alpha}\frac{|\tau_z|}{\sigma_0'}\right)^{1/\beta}} \qquad (13)$$

in which $|\tau_z|$ is the amplitude of the shear stress τ_z.

The pressures generated by the shear stress components, Eqs. (11)-(13), are essentially additive. Therefore, the pressure generated in the near field (the source term f) will be:

$$f = \frac{\sigma_0'}{T}\frac{1}{\left(\frac{1}{\alpha}\frac{|\tau_y|}{\sigma_0'}\right)^{1/\beta}} + \frac{\sigma_0'}{T}\frac{1}{\left(\frac{1}{\alpha}\frac{|\tau_x|}{\sigma_0'}\right)^{1/\beta}} + \frac{\sigma_0'}{T}\frac{1}{\left(\frac{1}{\alpha}\frac{|\tau_z|}{\sigma_0'}\right)^{1/\beta}} \qquad (14)$$

Now, the pressure generated in the far field is a 1D process, discussed in the previous section. Eq. (14) reduces to that given for the 1D case, Eq. (9), when $\tau_x \to 0$ and $\tau_z \to 0$ in the far field.

5. Comparison with the work of others

Table 1 compares extensions of 1D equation governing buildup of pore pressure to 3D by various authors (Sumer and Cheng, 1999, Li and Jeng, 2008, and Sui et al., 2019) including the present work.

Note that, for easy reference to Table 1, the Biot notation for the stress components (which Sumer, 2014, adopted in his book, and also in the present note, Fig. 2.b) and the more familiar notation for the same quantities are displayed in the following.

Stresses in Biot notation	Stresses in more familiar notation
$\begin{pmatrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{pmatrix}$	$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$

Table 1. Comparison of extension of 1D equation governing buildup of pore pressure to 3D by various authors.

Study	Diffusion terms in Eq. (3)	Source term in Eq. (3), f
Sumer and Cheng (1999)	$c_v(\frac{\partial^2 \overline{p}}{\partial x^2}+\frac{\partial^2 \overline{p}}{\partial z^2})$, 2D case	$f=\frac{\sigma_0'}{N_\ell T}(\frac{t}{N_\ell T})^s$ (15) with N_ℓ calculated from 1D approach, Eq. (8), with no regard to the presence of the structure
Li and Jeng (2008)	$c_v(\nabla^2 \overline{p})$, 3D case	Eqs. (7) and (8) in the present note with τ $\tau=\frac{1}{3}(\lvert\tau_{rz}\rvert+\lvert\tau_{r\theta}\rvert+\lvert\tau_{\theta z}\rvert)$ (16)
Sui et al. (2019)	$c_v(\nabla^2 \overline{p})$, 3D case	Eqs. (7) and (8) in the present note with τ $\tau=(\tau_x{}^2+\tau_y{}^2+\tau_z{}^2)^{1/2}$ (17)
Present study	Eqs. (3)-(6) in the present note	Eq. (14) in the present note

[1] The quantity s in Sumer and Cheng (1999) is a shape factor, and was chosen as $s=-0.82$.
[2] The amplitude of the shear stress in Li and Jeng (2008) is taken as the mean of the amplitudes of three components of shear stresses, as indicated in the table. Li and Jeng (2008) used cylindrical coordinates in their study.

Extension of the 1D equation governing the buildup of pore pressure to 2D or 3D is straightforward for the diffusion term in Eq. (3), using the diffusion analogy (Sumer, 2014, p. 74), as mentioned previously, and from Table 1, there is evidently consensus between all the authors.

This is not so, however, when the source term f in Eq. (3) is considered. As mentioned earlier, Sumer and Cheng (1999) are the first to extend the 1D equation to multi-dimensions (2D in their case) in the case where a pipeline is buried in the soil, subject to a progressive wave. The source term f in Sumer and Cheng (1999), however, was calculated through the 1D approach, Eq. (8), with no regard to the presence of the pipeline structure. Therefore, the 2D effect is sought in terms of the spread of the accumulated pore pressure only through the diffusion terms in Eq. (3). Sumer and Cheng (1999) used the so-called random-walk model to numerically simulate the process, and they compared their simulation results with laboratory experiments. They obtained some agreement with the experiments. However, the agreement was not perfect, implying that the use of the 1D approach in calculating the source term f in a 3D situation was not entirely justified.

Regarding the second study in Table 1, Li and Jeng (2008), these authors presumably take the amplitude of the shear stress in Eq. (9) as the mean of the amplitudes of shear stresses in the 3D case. One shortfall of this model is that the amplitude of the shear stress, Eq. (16), does not reduce to that of the 1D case in the far field when the other two components $\tau_{rz} \to 0$ and $\tau_{r\theta} \to 0$, but rather it reduces to $\tau = \frac{1}{3}(|\tau_{rz}|)$, one third of the amplitude of the shear stress of the 1D case.

As for the third study in Table 1, Sui et al. (2019), Eq. (17) reduces correctly to the familiar 1D case when $\tau_x \to 0$ and $\tau_z \to 0$ in the far field, as opposed to the shortfall experienced in the Li and Jeng model in this regard. However, it is expected that, in the process of buildup of pore pressure, each shear stress acts independently in re-arranging the soil grains (and therefore making the pore pressure accumulate), rather than the resultant shear stress controlling the buildup process.

Overall, it is felt that the present model, Eq. (14), describes the actual process closer to what happens in reality, compared with the previously proposed models.

6. How to proceed in the calculation of buildup of pore pressure in 3D? The procedure

1. Determine the amplitudes ($|\tau_y|$, $|\tau_x|$, and $|\tau_z|$) of the shear stress components in the soil in the presence of the structure, τ_x, τ_y and τ_z, using the 3D Biot equations in Open Foam, or any other 3D solver/code. (Analogous to 1D case, for example, like in the case of a progressive wave travelling over a soil of infinitely large depth with no structure present studied in Sumer (2014, Section 2.2.1) with the solution for the shear stress like $\tau_y = -ip_b \lambda z \exp(-\lambda z) \exp\left[i(\lambda x + \omega t)\right]$.)

2. Insert the calculated amplitudes of these shear stresses in the expression in Eq. (14), to calculate the source term f.

3. Subsequently, insert the calculated f into Eq. (3)-(5), to calculate the buildup of pore pressure.

4. In the case of the NuLIMAS (http://nulimas.info/) structure, GICON-SOF, a gravity structure, a 3D case (https://www.gicon.de/en/geschaeftsbereiche/gte/sf.html), this is straightforward.

5. For the interesting pipeline problem, liquefaction around a pipeline buried in the soil, cited previously in conjunction with Sumer and Cheng (1999) study, do the same exercise, namely, calculate τ_y (2D problem!), using Open Foam Biot Equations, or any other solver/code, and then insert the latter into Eq. (14), and subsequently insert the calculated f into Eq. (3)-(5), to calculate the buildup of pore pressure, just as in the case of the gravity structure above.

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